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The quantum damped driven harmonic oscillator

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Abstract. Using the Caldirola-Kanai Hamiltonian with an external driving force for the damped driven harmonic oscillator as the quantum dissipative system, we have exactly evaluated the propagator, wavefunction, uncertainty relation and the transition amplitudes by the Feynman path integral method.

1. Introduction

Although the Feynman path integral formulation (Feynman and Hibbs 1965) offers a general approach for treating quantum mechanical systems, only a few time-dependent Schrödinger equations can be solved exactly. One of these solvable problems is the damped harmonic oscillator described by the Caldiróla-Kanai Hamiltonian (Caldirola 1941, 1983, Kanai 1948). One can obtain the time-dependent Schrödinger equation for the damped harmonic oscillator by replacing the momentum with $(\hbar/\mathrm{i})(\partial/\partial x)$ in the Caldirola-Kanai Hamiltonian. However, the question is whether or not this equation represents the quantum mechanical dissipative system (Dekker 1981) as the Caldirola-Kanai Hamiltonian does for a classical case. Some workers (Dodonov and Manko 1979) have claimed that the Caldirola-Kanai equation describes the quantum dissipative systems, while others (Greenberger 1979, Senitzky 1960) have objected to it. The main flaw of the Dodonov-Manko result is its uncertainty relation, $\Delta p \Delta x \geqslant \mathrm{e}^{-\alpha t}\hbar/2$ in which the uncertainty vanishes as $t \to \infty$. This difficulty is critically reviewed by Greenberger (1979) and Cervero and Villarroel (1984). Greenberger has introduced the variable mass: $m(t) = m_0\,\mathrm{e}^{\alpha t}$ and removed the violation of the uncertainty.

The purpose of this paper is to extend our previous results for the damped harmonic oscillator (Yeon *et al* 1982, 1985a, b) to the damped driven harmonic oscillator by the path integral method. We introduce the Caldirola-Kanai Hamiltonian with an external driving force for the damped driven harmonic oscillator. We review the classical case of the Caldirola-Kanai Hamiltonian with an external driving force as a model for the time-dependent harmonic oscillator in § 2. In § 3 we first evaluate the propagator for the damped free particle and then the propagator for the damped quadratic Hamiltonian system. Section 4 gives the exact derivation of the propagator for the damped driving harmonic oscillator. In § 5 we evaluate the wavefunction by using the results obtained in § 4. The energy expectation values at various states are given in § 6. In § 7 without any ambiguity we evaluate explicitly the uncertainty relation at various states and show that it does not vanish as $t \to \infty$, but oscillates. Section 8 gives an explicit formula for the transition amplitudes and probabilities. In § 9 we discuss our results with examples.

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2. Classical case

We introduce the Hamiltonian of the time-dependent damped driven harmonic oscillator (DDHO) as

$$H = \mathrm{e}^{-\alpha t} p^2/2m + \mathrm{e}^{\alpha t}(\tfrac{1}{2} m\omega_0^2 x^2 - xf(t)) \tag{2.1}$$

where $f(t)$ is an external driving force and α is the positive constant. The Hamilton equations of motion for (2.1) are

$$\dot{x} = \mathrm{e}^{-\alpha t} p/m \tag{2.2}$$

$$\dot{p} = -\mathrm{e}^{+\alpha t}(m\omega_0^2 x - f(t)). \tag{2.3}$$

Equations (2.1) and (2.2) yield the Lagrangian

$$L = \mathrm{e}^{\alpha t}(\tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} m\omega_0^2 x^2 + xf(t)) \tag{2.4}$$

and the corresponding equation of motion is

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x^2 = f(t)/m. \tag{2.5}$$

Equation (2.1) can be considered as the Hamiltonian of a quantum damped driven harmonic oscillator, which bears analogy with that of a classical damped driven harmonic oscillator. The classical solution of equation (2.5) is

$$x(t) = A\,\mathrm{e}^{-\alpha t/2}\cos(\omega t + \varphi) + \mathrm{e}^{-\alpha t}\cos\omega t \int^{t} \cos 2\omega t'\,\mathrm{d}t' \int^{t'} \frac{f(s)}{m}\,\mathrm{e}^{-\alpha s/2}\cos\omega s\,\mathrm{d}s \tag{2.6}$$

with

$$\omega = (\omega_0^2 - \alpha^2/4)^{1/2}. \tag{2.7}$$

The mechanical energy can be expressed as

$$E = \mathrm{e}^{-2\alpha t} p^2/2m + \tfrac{1}{2} m\omega_0^2 x^2. \tag{2.8}$$

Here, the energy expression in equation (2.8) is not equal to the Hamiltonian itself.

3. Path integral of the oscillator system

In path integral formulation the solution of the Schrödinger equation is given as the path-dependent integral equation with propagator K:

$$\psi(x, t) = \int K(x, t; x_0, 0)\psi(x_0, 0)\,\mathrm{d}x_0 \tag{3.1}$$

which gives the wavefunction $\psi(x, t)$ at time t in terms of the wavefunction $\psi(x_0, 0)$ at time $t = 0$. The propagator can be expressed by the path integral

$$K(x, t; x_0, 0) = \lim_{N\to\infty} \int_{x(0)=x_0}^{x(t)=x} \exp\left(\frac{\mathrm{i}}{\hbar} S_{\mathrm{c}}(x, x_0, t)\right) \prod_{j=1}^{N-1} \frac{\mathrm{d}x_j}{A_j} \tag{3.2}$$

where the integrations are over all possible paths between two points and $S_c(x, x_0, t)$ is the classical action defined as the integral of the Lagrangian $L(x, x_0, t')$ between $t=t$ and $t=0$:

$$S_c(x, x_0, t) = \int_0^t L(x, x_0, t')\,\mathrm{d}t'. \tag{3.3}$$

In (3.2) A_j is the normalisation factor given by

$$A_j = \left(\frac{m}{2\pi \mathrm{i}\hbar\varepsilon \exp[-\alpha(t_j' + \varepsilon/2)]}\right)^{1/2} \qquad \varepsilon = \frac{t}{N}. \tag{3.4}$$

Since the Hamiltonian of a damped free particle is

$$H = \mathrm{e}^{-\alpha t} p^2/2m \tag{3.5}$$

and the corresponding Lagrangian becomes

$$L = \mathrm{e}^{\alpha t}\tfrac{1}{2}m\dot{x}^2. \tag{3.6}$$

Substituting equations (3.3)-(3.6) into (3.2), we obtain the damped free particle propagator:

$$K(x, t; x_0, 0) = \left(\frac{\alpha m\,\mathrm{e}^{\alpha t/2}}{4\pi \mathrm{i}\hbar \sinh \frac{1}{2}\alpha t}\right)^{1/2} \exp\left(\frac{\mathrm{i}\alpha m\,\mathrm{e}^{\alpha t/2}}{4\hbar \sinh\frac{1}{2}\alpha t}(x - x_0)^2\right). \tag{3.7}$$

Now we return to the Hamiltonian and the Lagrangian of the damped harmonic oscillator. If the Lagrangian is quadratic, then the propagator can be written as

$$K(x, t; x_0, 0) = F(t) \exp\left(\frac{\mathrm{i}}{\hbar} S_c(x, x_0, t)\right). \tag{3.8}$$

The propagator K can be determined explicitly if one evaluates the multiplicative function $F(t)$ given in the form

$$F(t) = \int_0^t \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{e}^{\alpha t}(\tfrac{1}{2}m\dot{y}^2 - \tfrac{1}{2}m\omega_0^2 y^2)\right) \prod_{j=1}^{N-1} \frac{\mathrm{d}y_j}{A_j} \tag{3.9}$$

where $y(t')$ is the deviation of $x(t')$ from its classical limit and all paths $y(t')$ arrive at $(0, t)$ from $(0, 0)$. Thus these paths can be expressed as a damped Fourier sine series with a fundamental period t:

$$y(t') = \sum_{n=1}^{N} a_n\,\mathrm{e}^{-\alpha t'/2} \sin\left(\frac{n\pi}{t}t'\right) \qquad t' = \varepsilon N. \tag{3.10}$$

Substituting (3.10) into (3.9) and integrating over t' the path integral $F(t)$ becomes

$$F(t) = \prod_{n=1}^{N} \left\{\varepsilon \exp\left[-\alpha\left(n\varepsilon + \frac{\varepsilon}{2}\right)\right]\left(\frac{n\pi}{t}\right)^2\right\}^{-1/2} \prod_{n=1}^{N}\left(1 - \frac{\omega_0^2 - (\alpha/2)^2}{n^2\pi^2}t^2\right)^{-1/2}. \tag{3.11}$$

As $N \to \infty$ and $\varepsilon \to 0$, the second bracket has the limit sinusoidal form. Thus we may write (3.11) as

$$F(t) = C\left(\frac{\sin \omega t}{\omega t}\right)^{-1/2} \tag{3.12}$$

where ω is given in (2.7). Since (3.12) should be reduced to (3.7) for $\omega_0 = 0$, we can determine the constant C. The path integral $F(t)$ becomes

$$F(t) = \left(\frac{m\omega\, \mathrm{e}^{\alpha t/2}}{2\pi \mathrm{i}\hbar \sin \omega t}\right)^{1/2}. \tag{3.13}$$

Hence the propagator of the damped quadratic Hamiltonian system can be written as

$$K(x, t; x_0, 0) = \left(\frac{m\omega\, \mathrm{e}^{\alpha t/2}}{2\pi \mathrm{i}\hbar \sin \omega t}\right)^{1/2} \exp\left(\frac{\mathrm{i}}{\hbar} S_c(x, x_0, t)\right). \tag{3.14}$$

Here the classical action has not explicitly been evaluated, but we will work it out in the next section.

4. Propagator

The classical action of the DDHO Hamiltonian is

$$S_c = \int \mathrm{e}^{\alpha t'}(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega_0^2 x^2 + xf(t'))\, \mathrm{d}t'. \tag{4.1}$$

In (4.1) for small ω_0 the kinetic energy is dominant and then the Lagrangian acts like that of a damped free particle under the driven force. Therefore we may give the propagator as

$$K(x, t; x_0, 0) = F(t) \exp[-a(\mathrm{e}^{\alpha t}x - x_0)^2]. \tag{4.2}$$

Here a is a time-dependent function. Since we know the Gaussian dependence of the propagator for a damped free particle, we may take the propagator for DDHO as having the following form:

$$K(x, t; x_0, 0) = \exp\left\{-\left[a(t)\, \mathrm{e}^{\alpha t}\frac{m\omega_0}{\hbar}x^2 + b(t)\, \mathrm{e}^{\alpha t/2}\left(\frac{m\omega_0}{\hbar}\right)^{1/2} x + c(t)\right]\right\}. \tag{4.3}$$

Changing variables t and x into

$$p = \frac{\omega_0}{2\mathrm{i}}t \qquad y = \left(\frac{m\omega_0}{\hbar}\right)^{1/2} x \tag{4.4}$$

the propagator is then expressed by

$$K(y, p; y_0, 0) = \exp\left\{-\left[a(p) \exp\left(\mathrm{i}\frac{2\alpha}{\omega_0}p\right)y^2 + b(p) \exp\left(\mathrm{i}\frac{\alpha}{\omega_0}p\right)y + c(p)\right]\right\} \tag{4.5}$$

and (4.5) must satisfy the Schrödinger equation

$$\mathrm{i}\hbar\, \partial K/\partial t = HK. \tag{4.6}$$

Substitution of (4.5) and (2.1) into (4.6) gives the time-dependent coefficients

$$a(t) = \beta + \tfrac{1}{2}\gamma \cot(2\gamma p) \tag{4.7}$$

with

$$\beta = \mathrm{i}\frac{\alpha}{4\omega_0} \qquad \gamma = \left(\frac{\alpha^2}{4\omega_0^2} - 1\right)^{1/2} \tag{4.8}$$

$$b(t) = \frac{B}{\sin(2\gamma p)} - \frac{2(m\hbar\omega_0^3)^{1/2}}{\sin(2\gamma p)} \int^p f(p) \exp\left(\mathrm{i}\frac{\alpha}{\omega_0} p\right) \sin(2\gamma p)\, \mathrm{d}p \tag{4.9}$$

and

$$\begin{aligned} c(t) = &-\frac{B^2}{2\gamma^2} - 2\beta p + \tfrac{1}{2}\log[\sin(2\gamma p)] \\ &+ \int^p \left\{ \frac{1}{\sin^2(2\gamma p)} \left[\frac{2}{(m\hbar\omega_0^3)^{1/2}} \int^p f(p) \exp\left(\frac{2\alpha}{\omega_0} p\right) \sin(2\gamma p)\, \mathrm{d}p \right]^2 \right. \\ &\left. - \frac{2B}{\sin^2(2\gamma p)} \frac{2}{(m\hbar\omega_0^3)^{1/2}} \int^p f(p) \exp\left(\mathrm{i}\frac{\alpha}{\omega_0} p\right) \sin(2\gamma p)\, \mathrm{d}p \right\} \mathrm{d}p + \log c_0 \end{aligned} \tag{4.10}$$

where the constants B and c_0 obtained in comparison with (4.2), (4.7), (4.9) and (4.10) can be expressed as

$$B = \mathrm{i}\left(\frac{m\omega^2}{\hbar\omega_0}\right)^{1/2} x_0 \tag{4.11}$$

$$\log c_0 = \log\left(\frac{m\omega}{2\pi\mathrm{i}\hbar}\right)^{1/2} - \mathrm{i}\frac{m}{4\hbar} x_0^2. \tag{4.12}$$

Substitution of (4.7)-(4.12) into (4.3) gives the propagator for DDHO:

$$\begin{aligned} K(x, t; x_0, 0) = &\left(\frac{m\omega\, \mathrm{e}^{\alpha t/2}}{2\pi\mathrm{i}\hbar \sin\omega t}\right)^{1/2} \exp\frac{\mathrm{i}m}{4\hbar}\left(\alpha(x_0^2 - \mathrm{e}^{\alpha t}x^2)\right. \\ &+ \frac{2\omega}{\sin\omega t}[(x^2\mathrm{e}^{\alpha t} + x_0^2)\cos\omega t - 2\,\mathrm{e}^{\alpha t/2} x_0 x] \\ &+ \frac{4x\,\mathrm{e}^{\alpha t/2}}{m\sin\omega t}\int_0^t f(t')\,\mathrm{e}^{\alpha t'/2}\sin\omega t'\,\mathrm{d}t' \\ &+ \frac{4x_0}{m\sin\omega t}\int_0^t f(t')\,\mathrm{e}^{\alpha t'/2}\sin\omega(t-t')\,\mathrm{d}t' \\ &\left.+ \frac{1}{m^2\omega}\int_0^t\int_0^{t'} f(t')f(S)\exp[\alpha(t'+S)/2]\sin\omega(t-t')\sin\omega S\,\mathrm{d}S\,\mathrm{d}t'\right). \end{aligned} \tag{4.13}$$

Setting $f(t) = 0$ or $\alpha = 0$, (4.13) is reduced to the propagator of the damped harmonic oscillator or the forced harmonic oscillator. When $\alpha = f(t) = 0$, (4.13) becomes the familiar propagator of the harmonic oscillator. To simplify the expression we write (4.13) as

$$K(x, t; x_0, 0) = F(t)\exp\left(\frac{\mathrm{i}m}{2\hbar}(\tilde{a}x^2 + \tilde{b}x_0^2 + 2\tilde{c}x_0x + 2\tilde{d}x + 2\tilde{e}x_0 - \tilde{f})\right) \tag{4.14}$$

where

$$\tilde{a} = (-\tfrac{1}{2}\alpha + \omega \cot \omega t)\, e^{\alpha t} \tag{4.15}$$

$$\tilde{b} = (\tfrac{1}{2}\alpha + \omega \cot \omega t) \tag{4.16}$$

$$\tilde{c} = -\frac{\omega}{\sin \omega t}\, e^{\alpha t/2} \tag{4.17}$$

$$\tilde{d} = \frac{\Delta(t)}{m \sin \omega t}\, e^{\alpha t/2} \tag{4.18}$$

$$\tilde{e} = \frac{\Box(t)}{m \sin \omega t} \tag{4.19}$$

$$\Delta(t) = \int_0^t f(t')\, e^{\alpha t'/2} \sin \omega t' \, dt' \tag{4.20}$$

$$\Box(t) = \int_0^t f(t')\, e^{\alpha t'/2} \sin \omega(t - t') \, dt' \tag{4.21}$$

$$\tilde{f} = \frac{\nabla(t)}{2m^2\omega} \tag{4.22}$$

$$\nabla(t) = \int_0^t \int_0^{t'} f(t') f(S) \exp[\alpha(S + t')/2] \sin \omega(t - t') \sin \omega S \, dS \, dt' \tag{4.23}$$

$$F(t) = \left(\frac{m\omega\, e^{\alpha t/2}}{2\pi i \hbar \sin \omega t}\right)^{1/2}. \tag{4.24}$$

5. Wavefunction

In this section we will discuss the wavefunction of DDHO. The Hamiltonian of DDHO (equation (2.1)) reduces to the quadratic form at $t = 0$:

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\omega_0^2 \left(x - \frac{f(0)}{m\omega_0^2}\right)^2 - \frac{f^2(0)}{2m\omega_0^2}. \tag{5.1}$$

Changing the variable x into $x + f^2(0)/m\omega_0^2$ we can eliminate the constant term in (5.1). The corresponding wavefunction $\psi_n(x, 0)$ and the energy eigenvalue are given by

$$\psi_n(x, 0) = N_0 H_n \left[\alpha_0 \left(x - \frac{f(0)}{m\omega_0^2}\right)\right] \exp\left[-\tfrac{1}{2}\alpha_0^2 \left(x - \frac{f(0)}{m\omega_0^2}\right)^2\right] \tag{5.2}$$

$$E_n = (n + \tfrac{1}{2})\hbar\omega - \frac{f^2(0)}{2m\omega_0^2} \tag{5.3}$$

where H_n is the Hermite polynomial of order n and the coefficients are

$$\alpha_0 = \left(\frac{m\omega}{\hbar}\right)^{1/2} \qquad N_0 = \frac{\alpha_0^{1/2}}{(2^n n! \sqrt{\pi})^{1/2}}. \tag{5.4}$$

For the convenience of the calculation of other quantities we set $f(0)=0$, and then (5.2) and (5.3) reduce to the following:

$$\psi_n(x,0)=N_0H_n(\alpha_0x)\exp(-\tfrac{1}{2}\alpha_0x^2) \tag{5.5}$$

$$E_n=(n+\tfrac{1}{2})\hbar\omega_0. \tag{5.6}$$

With the use of (3.1), (4.14) and (5.5), the wavefunction can be obtained in the form

$$\begin{aligned}\psi_n(x,t)&=\int_{-\infty}^{\infty}K(x,t;x_0,0)\psi_n(x_0)\,\mathrm{d}x_0\\&=N_0F(t)\exp\left(\frac{\mathrm{i}m}{2\hbar}(\tilde{a}x^2+2\tilde{d}x-\tilde{f})\right)\\&\quad\times\int_{-\infty}^{\infty}\mathrm{d}x_0\exp\left\{\frac{\mathrm{i}m}{2\hbar}\left[\left(\tilde{b}-\frac{\hbar\alpha_0}{\mathrm{i}m}\right)x_0^2+2(\tilde{c}x+\tilde{e})x_0\right]\right\}H_n(\alpha_0x_0).\end{aligned} \tag{5.7}$$

Using the following relations:

$$\int_{-\infty}^{\infty}\exp[-(x-y)^2]H_n(ax)\,\mathrm{d}x=\sqrt{\pi}(1-a^2)^{n/2}H_n[ay(1-a^2)^{-1/2}] \tag{5.8}$$

$$\left(\frac{a-\mathrm{i}b}{a+\mathrm{i}b}\right)^{n/2}=\exp\left(-\mathrm{i}n\tan^{-1}\frac{b}{a}\right)=\exp\left(-\mathrm{i}n\cot^{-1}\frac{a}{b}\right) \tag{5.9}$$

we finally obtain the wavefunction

$$\begin{aligned}\psi_n(x,t)=N\frac{1}{(2^nn!)^{1/2}}\exp\left\{-\mathrm{i}\left[(n+\tfrac{1}{2})\cot^{-1}\left(\frac{\alpha}{2\omega}+\cot\omega t\right)+\tilde{f}\right]\right\}\\\times\exp[-(Ax^2+2Bx)]H_n[D(x-E)].\end{aligned} \tag{5.10}$$

Here

$$N=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{\exp[-(\Gamma(t)-\frac{1}{4}\alpha t)]}{\zeta(t)(\sin\omega t)^{1/2}} \tag{5.11}$$

$$\zeta^2(t)=\frac{\alpha^2}{4\omega^2}+\frac{\alpha}{\omega}\cot\omega t+\operatorname{cosec}^2\omega t \tag{5.12}$$

$$\Gamma(t)=\frac{\square(t)^2}{2\hbar m\omega^2\zeta(t)^2\sin^2\omega t} \tag{5.13}$$

$$A(t)=\frac{m\omega}{2\hbar}\mathrm{e}^{\alpha t}\left[\frac{1}{\zeta(t)^2\sin^2\omega t}+\mathrm{i}\left(\frac{\alpha}{2\omega}-\cot\omega t+\frac{\alpha/2\omega+\cot\omega t}{\zeta(t)^2\sin^2\omega t}\right)\right] \tag{5.14}$$

$$B(t)=-\frac{\mathrm{e}^{\alpha t/2}}{2\hbar\zeta(t)^2\sin^2\omega t}\left[\square(t)-\mathrm{i}\left(\zeta(t)^2\sin\omega t\cdot\Delta(t)+\frac{\alpha/2\omega+\cot\omega t}{\sin\omega t}\square(t)\right)\right] \tag{5.15}$$

$$D(t)=\frac{\alpha_0\,\mathrm{e}^{\alpha t/2}}{\zeta(t)\sin\omega t} \tag{5.16}$$

$$E(t)=\frac{\square(t)\,\mathrm{e}^{-\alpha t/2}}{m\omega}. \tag{5.17}$$

6. Energy expectation values

The mechanical energy of DDHO (equation (2.8)) can be expressed as the energy operator

$$E_{\mathrm{op}} = -\frac{\hbar^2}{2m} \mathrm{e}^{-2\alpha t} \frac{\partial^2}{\partial x^2} + \tfrac{1}{2} m \omega_0^2 x^2. \tag{6.1}$$

The quantum mechanical expectation values of E take the form

$$\langle E \rangle_{mn} = -\frac{\hbar^2}{2m} \mathrm{e}^{-2\alpha t} \left\langle \frac{\partial^2}{\partial x^2} \right\rangle_{mn} + \tfrac{1}{2} m \omega_0^2 \langle x^2 \rangle_{mn}. \tag{6.2}$$

To evaluate the energy expectation values $\langle E \rangle_{mn}$ we use the following wavefunction (see equation (5.10)):

$$\begin{aligned} \psi_n(x, t) = N \left(\frac{1}{2^n n!} \right)^{1/2} & \exp\left[-\mathrm{i}(n + \tfrac{1}{2}) \cot^{-1}\left(\frac{\alpha}{2\omega} + \cot \omega t \right) \right] \\ & \times \exp[-(Ax^2 + 2Bx)] H_n[D(x - E)]. \end{aligned} \tag{6.3}$$

In (6.3) we have eliminated the imaginary part of f, which does not include n and x in the exponent, because it does not contribute to the probability or the expectation values of the physical quantities.

The expectation value of x^2, i.e. $\langle x^2 \rangle_{mn}$, is

$$\begin{aligned} \langle x^2 \rangle_{mn} &= \int \psi_m^*(x, t) x^2 \psi_n(x, t) \, \mathrm{d}x \\ &= \left(\frac{1}{2^{n+m} \pi n! \, m!} \right)^{1/2} \exp[-\mathrm{i}(n - m) \cot^{-1}(\alpha/2\omega + \cot \omega t)] \\ &\quad \times \left[\frac{1}{2 \operatorname{Re} A} \int_{-\infty}^{\infty} \mathrm{e}^{-y^2} y^2 H_m(y) H_n(y) \, \mathrm{d}y \right. \\ &\quad + \left(\frac{\operatorname{Re} B}{\operatorname{Re} A} \right)^2 \int_{-\infty}^{\infty} \mathrm{e}^{-y^2} H_m(y) H_n(y) \, \mathrm{d}y \\ &\quad \left. - \frac{\sqrt{2} \operatorname{Re} B}{(\operatorname{Re} A)^{3/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-y^2} y H_m(y) H_n(y) \, \mathrm{d}y \right]. \end{aligned} \tag{6.4}$$

Here we changed the variable x into $x = y/(2 \operatorname{Re} A)^{1/2} - \operatorname{Re} B/\operatorname{Re} A$. Integrating over x we obtain the expression for $\langle x \rangle_{mn}$:

$$\begin{aligned} \langle x^2 \rangle_{mn} &= [(n+2)(n+1)]^{1/2} \frac{\hbar \zeta(t)^2 \sin^2 \omega t}{2m\omega \, \mathrm{e}^{\alpha t}} \\ &\quad \times \exp[2\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)] \delta_{m,n+2} \\ &\quad + (n+1)^{1/2} \left(\frac{2\hbar}{m^3 \omega^3} \right)^{1/2} \zeta(t) \, \mathrm{e}^{-\alpha t} \square(t) \sin \omega t \\ &\quad \times \exp[\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)] \delta_{m,n+1} \\ &\quad + \frac{\mathrm{e}^{-\alpha t}}{m^2 \omega^2} [(n + \tfrac{1}{2}) m \hbar \omega \zeta(t)^2 \sin^2 \omega t + \square(t)^2] \delta_{m,n} \\ &\quad + \sqrt{n} \left(\frac{2\hbar}{m^3 \omega^3} \right)^{1/2} \zeta(t) \sin \omega t \square(t) \, \mathrm{e}^{-\alpha t} \exp[-\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)] \delta_{m,n-1} \\ &\quad + [n(n-1)]^{1/2} \frac{\hbar}{2m\omega} \zeta(t)^2 \sin^2 \omega t \, \mathrm{e}^{-\alpha t} \\ &\quad \times \exp[-2\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)] \delta_{m,n-2}. \end{aligned} \tag{6.5}$$

We may evaluate $\langle \partial^2/\partial x^2 \rangle_{mn}$ in a similar way:

$$\left\langle \frac{\partial^2}{\partial x^2} \right\rangle_{mn} = \int_{-\infty}^{\infty} \psi_m^*(x, t) \frac{\partial^2}{\partial x^2} \psi_n(x, t)\, \mathrm{d}x. \tag{6.6}$$

Transforming x into $(y/D+E)$ and using the recurrence relations of Hermite polynomials, we obtain

$$\begin{aligned} \frac{\partial^2 \psi_n(x, t)}{\partial x^2} &= \frac{N}{(2^n n!)^{1/2}} \exp[-\mathrm{i}(n+\tfrac{1}{2}) \cot^{-1}(\alpha/2\omega + \cot \omega t)] \bigg\{ \frac{A^2}{D^2} H_{n+2}(y) \\ &\quad + \frac{4D}{B}(B+AE) H_{n+1}(y) + 4\left[(B+AE)^2 - (n+\tfrac{1}{2})\left(A - \frac{A^2}{D^2}\right) \right] H_n(y) \\ &\quad + 8(B+AE)\left(\frac{A}{D} - D\right) n H_{n-1}(y) + 4n(n-1) \\ &\quad \times \left(D - \frac{A}{D}\right)^2 H_{n-2}(y) \bigg\} \exp(-\lambda(y)) \end{aligned} \tag{6.7}$$

where

$$\lambda(y) = \frac{A}{D^2}\left(y^2 + \frac{2D}{A}(B+AE) y + \frac{D^2 E}{A}(AE+2B) \right). \tag{6.8}$$

Substitution of (6.7) in (6.6) and integration over x yields

$$\begin{aligned} \left\langle \frac{\partial^2}{\partial x^2} \right\rangle_{mn} &= 2[(n+2)(n+1)]^{1/2} \exp[2\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)] \frac{A^2}{D^2} \delta_{m,n+2} \\ &\quad + 4[2(n+1)]^{1/2} \exp[\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)] \frac{A}{D}(B+AE) \delta_{m,n+1} \\ &\quad + 4\left[(n+\tfrac{1}{2})\left(\frac{A^2}{D^2} - A\right) + (B+AE)^2 \right] \delta_{m,n} \\ &\quad + 4(2n)^{1/2} \exp[-\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)]\left(\frac{A}{D} - D\right)(B+AE) \delta_{m,n-1} \\ &\quad + 2[n(n-1)]^{1/2} \exp[-\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)]\left(D - \frac{A}{D}\right)^2 \delta_{m,n-2}. \end{aligned} \tag{6.9}$$

Substituting (6.5) and (6.9) in (6.2), we can directly obtain the non-zero matrix elements of $\langle E \rangle_{mn}$ which occur only in the principal diagonal and the four diagonals adjacent to the principal diagonal:

$$\begin{aligned} \langle E \rangle_{n+2,n} &= [(n+2)(n+1)]^{1/2} \frac{\hbar\omega}{4} \mathrm{e}^{-\alpha t} \exp[2\mathrm{i} \cot^{-1}(\alpha/2\omega + \cot \omega t)] \\ &\quad \times \bigg\{ \zeta(t)^2 \sin^2 \omega t - \frac{1}{\zeta(t)^2 \sin^2 \omega t} + \frac{1}{\zeta(t)^2 \sin^2 \omega t} \\ &\quad \times \left[\left(\frac{\alpha}{2\omega} - \cot \omega t\right) \zeta(t)^2 \sin^2 \omega t + \frac{\alpha}{2\omega} + \cot \omega t \right]^2 \\ &\quad - 2\mathrm{i}\left[\left(\frac{\alpha}{2\omega} - \cot \omega t\right) \zeta(t)^2 \sin^2 \omega t + \frac{\alpha}{2\omega} + \cot \omega t \right] \bigg\} \\ &= [(n+2)(n+1)]^{1/2} \theta(t) \end{aligned} \tag{6.10}$$

$$\langle E\rangle_{n+1,n}=(n+1)^{1/2}\left(\frac{\hbar\omega}{2m}\right)^{1/2}\zeta(t)\sin\omega t\,\mathrm{e}^{-\alpha t}\exp[\mathrm{i}\cot^{-1}(\alpha/2\omega+\cot\omega t)]$$

$$\times\left\{\square(t)+\frac{1}{\zeta(t)^2\sin^2\omega t}\left[-\frac{\Delta(t)}{\sin\omega t}+\left(\frac{\alpha}{2\omega}-\cot\omega t\right)\square(t)\right]\right.$$

$$\left.\times\left[\left(\frac{\alpha}{2\omega}-\cot\omega t\right)\zeta(t)^2\sin^2\omega t+\frac{\alpha}{2\omega}+\cot\omega t-\mathrm{i}\right]\right\}$$

$$=(n+1)^{1/2}\eta(t) \tag{6.11}$$

$$\langle E\rangle_{nn}=(n+\tfrac{1}{2})\frac{\hbar\omega}{2}\mathrm{e}^{-\alpha t}\left(\zeta(t)^2\sin^2\omega t+\frac{1}{\zeta(t)^2\sin^2\omega t}\right)$$

$$+\frac{\square(t)^2}{2m}\mathrm{e}^{-\alpha t}+\frac{\mathrm{e}^{-\alpha t}}{2m}\left[-\frac{\Delta(t)}{\sin\omega t}+\square(t)\left(\frac{\alpha}{2\omega}-\cot\omega t\right)\right]^2 \tag{6.12}$$

$$\langle E\rangle_{n-1,n}=\sqrt{n}\,\eta(t)^* \tag{6.13}$$

and

$$\langle E\rangle_{n-2,n}=[n(n-1)]^{1/2}\theta(t)^*. \tag{6.14}$$

Except for the second off-diagonal elements ($E_{n+2,n}$ and $E_{n-2,n}$) the diagonal and the first off-diagonal elements are involved in the external driving force, i.e. $f(t)$.

7. The uncertainty relation

In this section we evaluate the uncertainty relation. In a similar way to that used to obtain (6.5) and (6.6) we calculate the quantities $\langle x\rangle_{mn}$ and $\langle\partial/\partial x\rangle_{mn}$:

$$\langle x\rangle_{mn}=\int_{-\infty}^{\infty}\psi_m^*(x,t)x\psi_n(x,t)\,\mathrm{d}x$$

$$=\frac{1}{2}\left(\frac{n+1}{\mathrm{Re}\,A}\right)^{1/2}\exp[\mathrm{i}\cot^{-1}(\alpha/2\omega+\cot\omega t)]\delta_{m,n+1}-\frac{\mathrm{Re}\,B}{\mathrm{Re}\,A}\delta_{m,n}$$

$$+\frac{1}{2}\left(\frac{n}{\mathrm{Re}\,A}\right)^{1/2}\exp[-\mathrm{i}\cot^{-1}(a/2\omega+\cot\omega t)]\delta_{m,n-1} \tag{7.1}$$

and

$$\left\langle\frac{\partial}{\partial x}\right\rangle_{mn}=\int_{-\infty}^{\infty}\psi_m^*(x,t)\frac{\partial}{\partial x}\psi_n(x,t)\,\mathrm{d}x$$

$$=[2(n+1)]^{1/2}\frac{A}{D}\exp[\mathrm{i}\cot^{-1}(\alpha/2\omega+\cot\omega t)]\delta_{m,n+1}-2(B+AE)\delta_{m,n}$$

$$-(2n)^{1/2}\left(\frac{A}{D}-D\right)\exp[-\mathrm{i}\cot^{-1}(\alpha/2\omega+\cot\omega t)]\delta_{m,n-1}. \tag{7.2}$$

With the help of (6.5) and (6.9) the uncertainty relations in the various states can be obtained:

$$[(\Delta x)^2(\Delta p)^2]^{1/2}_{n+2,n} = [(\langle x^2\rangle - \langle x\rangle^2)(\langle p^2\rangle - \langle p\rangle^2)]^{1/2}_{n+2,n}$$

$$= [(n+2)(n+1)]^{1/2}\frac{\hbar}{2}\left[\left(\frac{\alpha}{2\omega} - \cot\omega t\right)\zeta(t)^2\sin^2\omega t + \left(\frac{\alpha}{2\omega} + \cot\omega t\right) - \mathrm{i}\right]\exp[2\mathrm{i}\cot^{-1}(\alpha/2\omega + \cot\omega t)]$$

$$= [(n+2)(n+1)]^{1/2}\xi(t) \tag{7.3}$$

$$[(\Delta x)^2(\Delta p)^2]^{1/2}_{n+1,n}$$

$$= \hbar\Bigg[\!\!\Bigg[(2n)^{1/2}\frac{\zeta(t)\sin\omega t}{(m\hbar\omega)^{1/2}}\left[\left(\frac{\alpha}{2\omega} - \cot\omega t\right)\zeta(t)^2\sin^2\omega t + \left(\frac{\alpha}{2\omega} + \cot\omega t\right) - \mathrm{i}\right]\left[\frac{-\Delta(t)}{\sin\omega t} + \left(\frac{\alpha}{2\omega} - \cot\omega t\right)\square(t)\right]$$

$$+ \tfrac{1}{2}n\left\{1 - \left[\left(\frac{\alpha}{2\omega} - \cot\omega t\right)\zeta(t)^2\sin^2\omega t + \left(\frac{\alpha}{2\omega} + \cot\omega t\right)\right]^2 + 2\mathrm{i}\left[\left(\frac{\alpha}{2\omega} - \cot\omega t\right)\zeta(t)^2\sin^2\omega t + \left(\frac{\alpha}{2\omega} + \cot\omega t\right)\right]\right\}$$

$$\times\exp[\mathrm{i}\cot^{-1}(\alpha/2\omega + \cot\omega t)\Bigg]\!\!\Bigg]^{1/2}\left(\frac{[2(n+1)]^{1/2}\square(t)}{(m\hbar\omega)^{1/2}\zeta(t)\sin\omega t} - \tfrac{1}{2}n\right)^{1/2}$$

$$\times\exp[\mathrm{i}\cot^{-1}(\alpha/2\omega + \cot\omega t)] \tag{7.4}$$

$$[(\Delta x)^2(\Delta p)^2]^{1/2}_{n,n} = (n+\tfrac{1}{2})\hbar\left\{1 + \left[\left(\frac{\alpha}{2\omega} - \cot\omega t\right)\zeta(t)^2\sin^2\omega t + \left(\frac{\alpha}{2\omega} + \cot\omega t\right)\right]^2\right\}^{1/2} \tag{7.5}$$

and

$$[(\Delta x)^2(\Delta p)^2]^{1/2}_{n-2,n} = [n(n-1)]^{1/2}\xi(t)^*. \tag{7.6}$$

Taking the complex conjugate and changing $(n+1)$ into n in (7.4) we can easily obtain the uncertainty in the $(n-1, n)$ state.

8. Transition amplitudes

Using the wavefunction (equation (5.7)) we shall compute the transition amplitudes $a_{m,n}$ for the damped driven harmonic oscillator from a state $|m\rangle$ to a state $|n\rangle$. The transition amplitude is given by

$$a_{mn} = \int_{-\infty}^{\infty}\mathrm{d}x\,\psi_m(x,0)\psi_n(x,t) \tag{8.1}$$

$$= \left(\frac{m\omega/\hbar}{2^{m+n}\pi m!\,n!}\right)^{1/2}\frac{1}{\zeta(t)(\sin\omega t)^{1/2}}\exp(\tfrac{1}{4}\alpha t - \Gamma(t))$$

$$\times\exp[-\mathrm{i}(n+\tfrac{1}{2})\cot^{-1}(\alpha/2\omega + \cot\omega t)]$$

$$\times\int_{-\infty}^{\infty}\exp[-(\tfrac{1}{2}\alpha_0^2 + A)x^2 - 2Bx]H_m(\alpha_0 x)H_n[D(x-E)]\,\mathrm{d}x. \tag{8.2}$$

To evaluate the integral in (8.2) we define the generating function $S(x, s)$ and $S(x, t)$ for the nth-order Hermite polynomials as

$$S(x, s) = \sum_{m=0}^{\infty} \frac{H_m(\alpha_0 x)}{m!} s^m = \exp(-s^2 + 2\alpha_0 xs) \tag{8.3}$$

$$S(x, t) = \sum_{n=0}^{\infty} \frac{H_n[D(x-E)]}{n!} t^n = \exp[-t^2 + 2D(x-E)t]. \tag{8.4}$$

Multiplying (8.3) by (8.4) and the exponential term given in (8.2) we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{s^m t^m}{m!n!} \int_{-\infty}^{\infty} &\mathrm{d}x\, H_m(\alpha_0 x) H_n[D(x-E)] \exp[-(\tfrac{1}{2}\alpha_0^2 + A)x^2 - 2Bx] \\ &= \exp\Big(-(s^2+t^2) - 2DEt \int_{-\infty}^{\infty} \mathrm{d}x \exp\{-[(\tfrac{1}{2}\alpha_0^2 + A)x^2 \\ &\quad - 2Bx + 2s\alpha_0 x + 2tDx]\}\Big).\end{aligned} \tag{8.5}$$

Performing the integral for the right-hand side of (8.5) we obtain

$$\begin{aligned}\left(\frac{\pi}{\frac{1}{2}\alpha_0^2 + A}\right)^{1/2} &\sum_{i,j,k,l,p} (-1)^{j+l} \frac{[(\frac{1}{2}\alpha_0 - A)/(\frac{1}{2}\alpha_0 + A)]^i [2\alpha_0 B/(\frac{1}{2}\alpha_0 + A)]^j}{i!j!} \\ &\times \frac{[(D^2 - \frac{1}{2}\alpha_0 - A)/(\frac{1}{2}\alpha_0 + A)]^k [(2B + \frac{1}{2}\alpha_0 E + AE)/(\frac{1}{2}\alpha_0 + A)]^l (\alpha_0 D)^p}{k!l!p!} \\ &\times s^{2i+j+p} t^{2k+l+p}.\end{aligned} \tag{8.6}$$

Then the integral part of (8.5) can be written as

$$\begin{aligned}\int_{-\infty}^{\infty} &\mathrm{d}x\, H_m(\alpha_0 x) H_n[D(x-E)] \exp\{-[(\tfrac{1}{2}\alpha_0^2 + A)x^2 + 2Bx]\} \\ &= \left(\frac{\pi}{\frac{1}{2}\alpha_0 + A}\right)^{1/2} \sum_{p=0}^{\min(m,n)} \sum_{j=0}^{\min(m,n)-p} \sum_{l=0}^{\min(m,n)-p} \frac{(-1)^{j+l} m!n!\alpha_0^p D^{l+p}}{j!l!p!} \\ &\quad \times \frac{(\frac{1}{2}\alpha_0 - A)^{(m-j-p)/2} (2\alpha_0 B)^j (D^2 - \frac{1}{2}\alpha_0 - A)^{(n-l-p)/2} (2B + \frac{1}{2}\alpha_0 E + AE)^l}{[\frac{1}{2}(m-j-p)]![\frac{1}{2}(m-l-p)]!(\frac{1}{2}\alpha_0 + A)^{(m+n+k+l-2p)/2}}.\end{aligned} \tag{8.7}$$

If $\min(m, n)$ is even (odd), $(j+p)$ and $(l+p)$ should be even (odd). Therefore the summation should be done only for the case of even (odd) l and j. Hence we obtain the transition amplitude as

$$\begin{aligned}a_{mn} = \left(\frac{\pi m\omega/\hbar}{2^{n+m} m!n!}\right)^{1/2} &\frac{\exp(\frac{1}{4}\alpha t - \Gamma(t))}{\zeta(t)[(\frac{1}{2}\alpha_0 + A)\sin\omega t]^{1/2}} \\ &\times \exp[-\mathrm{i}(n+\tfrac{1}{2})\cot^{-1}(\alpha/2\omega + \cot\omega t)] \\ &\times \sum_{p=0}^{\min(m,n)} \sum_{j=0}^{\min(m,n)-p} \sum_{l=0}^{\min(m,n)-p} \frac{(-1)^{j+l} m!n!\alpha_0^p D^{l+p}}{j!l!p!} \\ &\times \frac{(\frac{1}{2}\alpha_0 - A)^{(m-j-p)/2} (2\alpha_0 B)^j (D^2 - \frac{1}{2}\alpha_0 - A)^{(n-l-p)/2} (2B + \frac{1}{2}\alpha_0 E + AE)^l}{[\frac{1}{2}(m-j-p)]![\frac{1}{2}(n-l-p)]!(\frac{1}{2}\alpha_0 + A)^{(m+n+j+l-2p)/2}}.\end{aligned} \tag{8.8}$$

The transition probability p_{mn} corresponding to (8.8) is

$$P_{mn} = |a_{mn}|^2 = \left(\frac{m\omega/\hbar}{2^{m+n}m!\,n!}\,\frac{\pi}{\frac{1}{2}\alpha_0 + A}\right)\frac{\exp(\frac{1}{2}\alpha t - 2\Gamma(t))}{\zeta(t)^2 \sin\omega t}$$
$$\times\left(\sum_{p,p'=0}^{\min(m,n)}\ \sum_{j,j'=0}^{\min(m,n)-p \text{ or } p'}\ \sum_{l,l'=0}^{\min(m,n)-p \text{ or } p'}\right.$$
$$\times\frac{(-1)^{j+j'+l+l'}(m!\,n!)^2\alpha_0^{p+p'}D^{l+l'+p+p'}}{j!\,j'!\,l!\,l'!\,p!\,p'!}$$
$$\times\frac{(\frac{1}{2}\alpha_0 - A)^{m-(j+j'+p+p')/2}(2\alpha_0 B)^{j+j'}(D^2 - \frac{1}{2}\alpha_0 - A)^{n-(l+l'+p+p')/2}}{[\frac{1}{2}(m-j-p)]!\,[\frac{1}{2}(m-j'-p')]!\,[\frac{1}{2}(n-l-p)]!\,[\frac{1}{2}(n-l'-p')]!}$$
$$\left.\times\frac{(2B + \frac{1}{2}\alpha_0 E + AE)^{l+l'}}{(\frac{1}{2}\alpha_0 + A)^{m+n+(j+j'+l+l')/2-(p+p')}}\right). \tag{8.9}$$

9. Results and discussion

In this section we shall discuss the results obtained in the previous sections. The propagator (equation (4.17)) and the wavefunction (equation (5.10)) are of new form. When we take $f(t)=0$, the propagator is reduced to the same structure obtained by Cheng (1984) and others (Khandekar and Lawanda 1979, Janusis *et al* 1979). In the case of $f(t) \neq$ constant the wavefunction is similar to that of Dodonov and Manko. To calculate the propagator $K(x, t; x_0, t_0)$ in Feynman's path integral we should know the classical action, i.e. the classical Lagrangian, which gives the classical equation of motion. However, we can obtain the same classical equation of motion from many different classical actions (Caratu *et al* 1976, Havas 1973, Currie and Saletin 1966) and thus one may have many different propagators corresponding to the classical actions. Therefore we should recognise that our propagator requires that the Hamiltonian be identical to the energy of the system. In this sense the mechanical energy operator (equation (6.1)) is not identical to the Hamiltonian operator (equation (2.1)). Therefore we assume that this Hamiltonian represents the quantum mechanical dissipative system. The matrix elements in the energy eigenvalues, which are involved in dissipation, should be examined in detail for a physical system.

To quantise the energy we have used the energy operator equation (6.1). Here, the momentum operator represents the canonical momentum expressed by $(\hbar/\mathrm{i})(\partial/\partial x)$. The energy expectation values given in (6.16)-(6.20) contain the term $\mathrm{e}^{-\alpha t}$ and thus decay exponentially. The second off-diagonal elements ($E_{n+2,n}$ and $E_{n-2,n}$) depend only on the exponential decaying constant α. The rest of all elements are involved in both the constant α and the external driving force $f(t)$.

Figures 1 and 2 illustrate the decay of the energy eigenvalue $E_{nn}(t)$ when $f(t)=f_0\delta(t-t_0)$ (see the appendix). We note that the results of Dodonov and Manko (1979) can be obtained by taking the driving force as $f(t)=f_0\sin(\omega t+\phi)$.

Through the calculations (7.1)-(7.2), we obtained the exact uncertainty relations in (7.3)-(7.6) at various states. The uncertainty for (n, n) states with period π (equation (7.5)) is reduced to that of the harmonic oscillator at 180° and 0°. We also note that for $f(t)=0$, (7.3)-(7.6) become those of the damped harmonic oscillator.

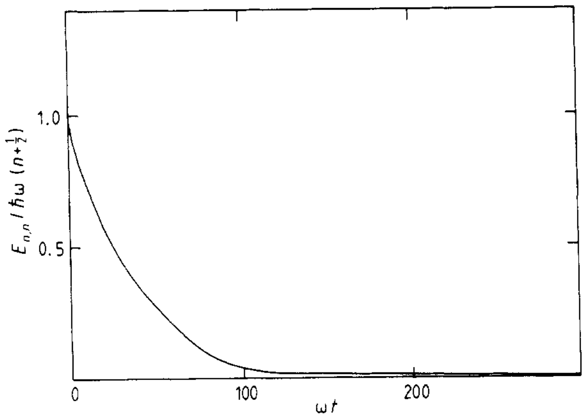

Figure 1. Energy eigenvalue $E_{n,n}(t)$. The curve represents the first term in (A3) at $\alpha/2\omega = 0.1$.

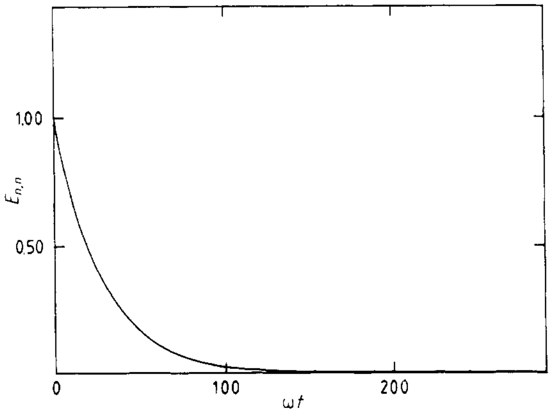

Figure 2. Energy eigenvalue $E_{n,n}(t)$. The curve represents the second term in (A3) at $\alpha/2\omega = 0.1$.

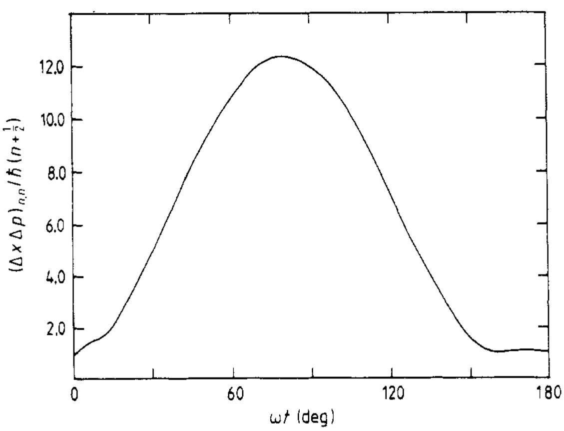

Figure 3. The uncertainty relation as the (n, n) state oscillates with period π.

For example the uncertainty for $(n-1, n)$ states is reduced to $[n(n-1)]^{1/2}\hbar$ and that for (n, n) states, $(n+\frac{1}{2})\hbar$. Figure 3 illustrates the uncertainty for (n, n) states under the driving force $f(t)=f_0\delta(t-t_0)$. It does not decay exponentially, but oscillates with period π and the uncertainty relation is satisfied.

The general expression for the transition amplitudes (equation (8.8)) and the transition probabilities (equation (8.9)) at various states are of a new form. Equation (8.9) is not zero and thus there exists the dissipative mechanism. Though the problems relating to the selection rules and the parity are not investigated in detail, we expect that the expressions for the amplitudes and the probabilities will be reduced to those obtained by Landovitz *et al* (1979, 1980, 1983a, b) for $f(t)=0$.

Appendix

As an example we take the delta function type for the external driving force:

$$f(t)=f_0\delta(t-t_0). \tag{A1}$$

Then the energy eigenvalue E_{nn} is obtained through the trivial calculation:

$$E_{nn}=(n+\tfrac{1}{2})\hbar\omega\tfrac{1}{2}\,\mathrm{e}^{-\alpha t}\left(\zeta(t)^2\sin^2\omega t+\frac{1}{\zeta(t)^2\sin^2\omega t}\right) \qquad t\leqslant t_0 \tag{A2}$$

$$\begin{aligned}E_{nn}=&(n+\tfrac{1}{2})\hbar\omega\tfrac{1}{2}\,\mathrm{e}^{-\alpha t}\left(\zeta(t)^2\sin^2\omega t+\frac{1}{\zeta(t)^2\sin^2\omega t}\right)\\&+\frac{f_0^2}{2m\omega^2}\exp[-\alpha(t-t_0)]\\&\times\left(1+\frac{\alpha^2}{4\omega^2}\sin^2\omega(t-t_0)+\frac{\alpha}{4\omega}\sin 2\omega(t-t_0)\right) \qquad t>t_0.\end{aligned} \tag{A3}$$

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